APPROXIMATION OF FUNCTIONS FROM THE CLASS $C_{\psi, \infty}^\beta$

BY POISSON INTEGRALS IN THE UNIFORM METRIC

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We obtain asymptotic equalities for upper bounds of deviations of the Poisson integrals on the class of continuous functions $C_{\psi, \infty}^\beta$ in the metric of the space $C$.

1. Statement of the Problem and Auxiliary Assertions

Let $f(\cdot)$ be a $2\pi$-periodic Lebesgue-summable function ($f \in L_1$). The Poisson integral of the function $f$ is introduced (see [1, p. 154] or [2, p. 161]) as the function $P(\rho; f; x)$ defined by the equality

$$P(\rho; f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t + x) \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} \rho^k \cos kt \right\} dt, \quad 0 \leq \rho < 1.$$ 

Setting $\rho = e^{-1/\delta}$, we represent the Poisson integral in the form

$$P_{\delta}(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t + x) \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} e^{-k/\delta} \cos kt \right\} dt, \quad \delta > 0.$$ 

In the present paper, we consider the class $C_{\psi, \infty}^\beta$ introduced by Stepanets (see, e.g., [3–6]), which is defined as follows: Assume that a function $f$ belongs to $L_1$ and its Fourier series has the form

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

Let $\psi(k)$ be an arbitrary function of a natural argument and let $\beta$ be a fixed real number. If the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left( a_k \cos \left( kx + \frac{\pi \beta}{2} \right) + b_k \sin \left( kx + \frac{\pi \beta}{2} \right) \right)$$

is the Fourier series of a certain function $\varphi \in L_1$, then $\varphi$ is called the $(\psi, \beta)$-derivative of the function $f$ and is denoted by $f_{\psi}^\beta(\cdot)$. Let $L_{\psi}^\beta$ denote the subset of all functions $f \in L_1$ that have $(\psi, \beta)$-derivatives. If $f$ belongs...
to $L^\psi_\beta$ and $f^\psi_\beta$ belongs to $\mathfrak{N}$, $\mathfrak{N} \subseteq L_1$, then one says that $f$ belongs to $L^\psi_{\beta, \mathfrak{N}}$. The subsets of continuous functions from $L^\psi_\beta$ and $L^\psi_{\beta, \mathfrak{N}}$ are denoted by $C^\psi_\beta$ and $C^\psi_{\beta, \mathfrak{N}}$, respectively. Further, if $\mathfrak{N}$ coincides with the unit ball of the space $L_\infty$, i.e.,

$$\mathfrak{N} = \{ f^\psi_\beta \in L_\infty : \text{ess sup}_t |f^\psi_\beta(t)| \leq 1 \},$$

then the classes $C^\psi_{\beta, \mathfrak{N}}$ are denoted by $C^\psi_{\beta, \infty}$.

In the present paper, we study the asymptotic behavior of the quantity

$$E \left( C^\psi_{\beta, \infty} ; P_\delta \right) = \sup_{f \in C^\psi_{\beta, \infty}} \| f(\cdot) - P_\delta(f(\cdot)) \|_C$$

(1)
as $\delta \to \infty$.

Following Stepanets [6, p. 198], we call the problem of finding asymptotic relations for quantity (1) as $\delta \to \infty$ the Kolmogorov–Nikol’skii problem for Poisson integrals on the class $C^\psi_{\beta, \infty}$ in the uniform metric.

Let $\mathfrak{M}$ denote the set of functions $\psi(\cdot)$ that satisfy the conditions

$$\mathfrak{M} = \{ \psi(t) : \psi(t) > 0, \psi(t_1) - 2\psi((t_1 + t_2)/2) + \psi(t_2) \geq 0 \ \forall t_1, t_2 \in [1, \infty), \ \lim_{t \to \infty} \psi(t) = 0 \}.$$ 

Let $\mathfrak{M}'$ denote the set of functions $\psi \in \mathfrak{M}$ for which

$$\int_1^\infty \frac{\psi(t)}{t} dt < \infty.$$

Using the characteristics

$$\eta(t) = \eta(\psi; t) = \psi^{-1} \frac{\psi(t)}{2}, \ \mu(t) = \mu(\psi; t) = \frac{t}{\eta(t) - t},$$

(2)

where $\psi^{-1}$ is the function inverse to $\psi$, one customarily considers (see, e.g., [5, p. 93] or [6, p. 160]) the following subsets of the set $\mathfrak{M}$:

$$\mathfrak{M}_0 = \{ \psi \in \mathfrak{M} : 0 < \mu(\psi; t) \leq K \ \forall t \geq 1 \},$$

$$\mathfrak{M}_C = \{ \psi \in \mathfrak{M} : 0 < K_1 < \mu(\psi; t) \leq K_2 \ \forall t \geq 1 \},$$

$$\mathfrak{M}_\infty = \{ \psi \in \mathfrak{M} : 0 < K \leq \mu(\psi; t) < \infty \ \forall t \geq 1 \}.$$

Here and in what follows, $K$ and $K_i$ denote constants, generally speaking, different in different relations and dependent on $\psi$. 
Note that, for functions $\psi \in \mathfrak{M}'_0$ ($\mathfrak{M}'_0 = \mathfrak{M}_0 \cap \mathfrak{M}'$) slowly decreasing to zero, i.e., for functions $\psi$ such that
\[
\int_1^\infty \psi(t) dt = \infty,
\]
the Kolmogorov–Nikol’skii problem was solved in [7]. The aim of the present paper is to find asymptotic equalities for upper bounds of deviations of Poisson integrals on the classes $C^\psi_\beta, C^\psi_1$ for $\beta \in R$ in the cases where $\psi \in \mathfrak{M}_C$ and $\psi \in \mathfrak{M}_\infty$, i.e., for functions $\psi(t)$ that decrease to zero as $t \to \infty$ faster than the function $1/t$, which determines the order of saturation of the linear approximation method generated by the operator $P_\delta$.

If the Fourier transform
\[
\hat{\tau}(t) = \hat{\tau}_\delta(t) = \frac{1}{\pi} \int_0^\infty \tau(u) \cos\left(ut + \frac{\beta \pi}{2}\right) du
\]
of the function $\tau(\cdot)$ defined by the equalities
\[
\tau(u) = \tau_\delta(u; \psi) = \begin{cases} 
(1 - e^{-u}) \frac{\psi(1)}{\psi(\delta)}, & 0 \leq u \leq \frac{1}{\delta}, \\
(1 - e^{-u}) \frac{\psi(\delta u)}{\psi(\delta)}, & u \geq \frac{1}{\delta},
\end{cases}
\]
is summable on the entire number axis, i.e., the integral $A(\tau)$
\[
A(\tau) = \int_{-\infty}^{\infty} |\hat{\tau}_\delta(t)| dt
\]
is convergent, then, for any $f \in C^\psi_\beta, C^\psi_1$, the following equality holds at every point $x \in R$:
\[
f(x) - P_\delta(f; x) = \psi(\delta) \int_{-\infty}^{+\infty} f_\beta^\psi\left(x + \frac{t}{\delta}\right) \hat{\tau}_\delta(t) dt, \quad \delta > 0.
\]
Note that, relation (6) can be obtained by repeating the arguments used in [6, p. 183]. Thus, to find asymptotic equalities for quantity (1) as $\delta \to \infty$ in the case where $\psi \in \mathfrak{M}_C$, $\psi \in \mathfrak{M}_\infty$, and $\beta \in R$, it is necessary to find conditions under which the Fourier transform $\hat{\tau}(t)$ is summable on the entire number axis.

2. Asymptotic Equalities for Upper Bounds of Deviations of Poisson Integrals from Functions of the Class $C^\psi_\beta, C^\psi_1$ in the Uniform Metric

The following statement is true:

**Theorem 1.** Suppose that $\psi \in \mathfrak{M}_C$, the function $g(u) = u \psi(u)$ is convex downward on $[b, \infty)$, $b \geq 1$, and
Then the following asymptotic equality holds as $\delta \to \infty$:

$$
\mathcal{E}\left(C_{\beta, \infty}^\psi; P_\delta\right)_C = \frac{1}{\delta} \sup_{f \in C_{\beta, \infty}^\psi} \left\| f_0^{(1)}(x) \right\|_C + O\left( \frac{1}{\delta^2} \int_1^\delta t \psi(t) dt + \frac{1}{\delta} \int_\delta^\infty \psi(t) dt \right),
$$

(8)

where $f_0^{(1)}$ is the $(\psi, \beta)$-derivative of the function $f$ for $\psi(t) = 1/t$ and $\beta = 0$.

Prior to the proof of Theorem 1, we consider the following lemma:

**Lemma 1.** Suppose that all conditions of Theorem 1 are satisfied. Then a Fourier transform $\hat{\tau}(t)$ of the form (3) for the function $\tau(u)$ defined by (4) is summable on the entire number axis, i.e., integral (5) is convergent.

**Proof of Lemma 1.** We set $\tau(u) = \varphi(u) + v(u)$, where

$$
\varphi(u) = \begin{cases} 
\frac{\psi(1)}{\psi(\delta)}, & 0 \leq u < \frac{1}{\delta}, \\
\frac{\psi(\delta u)}{\psi(\delta)}, & u \geq \frac{1}{\delta},
\end{cases}
$$

(9)

$$
v(u) = \begin{cases} 
(1 - e^{-u} - u)\frac{\psi(1)}{\psi(\delta)}, & 0 \leq u \leq \frac{1}{\delta}, \\
(1 - e^{-u} - u)\frac{\psi(\delta u)}{\psi(\delta)}, & u \geq \frac{1}{\delta},
\end{cases}
$$

(10)

and verify that the Fourier transforms

$$
\hat{\varphi}(t) = \hat{\varphi}_\delta(t) = \frac{1}{\pi} \int_0^\infty \varphi(u) \cos\left(ut + \frac{\beta \pi}{2}\right) du,
$$

(11)

$$
\hat{v}(t) = \hat{v}_\delta(t) = \frac{1}{\pi} \int_0^\infty v(u) \cos\left(ut + \frac{\beta \pi}{2}\right) du
$$

(12)

of the functions $\varphi$ and $v$, respectively, are summable on the entire number axis. Thus, it is necessary to show that the following integrals are convergent:

$$
A(\varphi) = \int_{-\infty}^\infty |\hat{\varphi}_\delta(t)| dt,
$$

(13)
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\[ A(v) = \int_{-\infty}^{\infty} |\hat{v}_\delta(t)| dt. \] 

(14)

First, we prove the convergence of integral (13). According to Theorem 1 in [8], for the convergence of the integral $A(\psi)$ it is necessary and sufficient that the following integrals be convergent:

\[
\begin{align*}
&\int_0^{1/2} u|d\psi'(u)|, \quad \int_{1/2}^{\infty} |u - 1||d\psi'(u)|, \\
&\left| \sin \frac{\beta \pi}{2} \right| \int_0^{\infty} \frac{|\psi(u)|}{u} du, \quad \int_0^{1/2} |\psi(1 - u) - \psi(1 + u)| du.
\end{align*}
\]

In follows from (9) that

\[ \varphi''(u) = 0, \quad u \in \left[0, \frac{1}{\delta}\right), \]

and

\[ \psi(\delta)|d\psi'u| \leq (2\delta|\psi'(\delta u)| + u\delta^2\psi''(\delta u))du, \quad \psi \in \mathcal{M}, \quad \text{for} \quad u \geq \frac{1}{\delta}. \] 

(15)

Since

\[ \int_0^{1/2} u|d\psi'(u)| = \int_{1/\delta}^{1/2} u|d\psi'(u)| \leq \int_{1/\delta}^{\infty} u|d\psi'(u)| \]

and

\[ \int_{1/2}^{\infty} |u - 1||d\psi'(u)| \leq \int_{1/\delta}^{\infty} u|d\psi'(u)|, \]

we obtain an estimate for the integral

\[ \int_{1/\delta}^{\infty} u|d\psi'(u)| \]

on each of the intervals $[1/\delta, b/\delta]$ and $[b/\delta, \infty)$ (for $\delta > 2b$). Taking (15) into account, we get

\[ \int_{1/\delta}^{b/\delta} u|d\psi'(u)| \leq \frac{2\delta}{\psi(\delta)} \int_{1/\delta}^{b/\delta} u|\psi'(\delta u)| du + \frac{\delta^2}{\psi(\delta)} \int_{1/\delta}^{b/\delta} u^2\psi''(\delta u) du. \]
Integrating both integrals on the right-hand side of the last inequality by parts and taking into account that \( \psi(\delta u) \leq \psi(1) \) for \( u \in [1/\delta, b/\delta] \), we get

\[
\int_{1/\delta}^{b/\delta} u |d\varphi'(u)| \leq \frac{K_1}{\delta \psi(\delta)}.
\]

Further, we show that the following relations are true:

\[
\lim_{u \to \infty} u \psi(u) = 0, \quad (16)
\]

\[
\lim_{u \to \infty} u^2 \psi'(u) = 0. \quad (17)
\]

Since the function \( g(u) = u \psi(u) \) is convex downward for \( u \geq b \geq 1 \), the following cases are possible: either

\[
\lim_{u \to \infty} g(u) = 0,
\]

or

\[
\lim_{u \to \infty} g(u) = K > 0,
\]

or

\[
\lim_{u \to \infty} g(u) = \infty.
\]

Let

\[
\lim_{u \to \infty} g(u) = K > 0.
\]

Then there exists \( 0 < K_1 < K \) such that, for all \( u \geq 1 \), one has \( g(u) > K_1 \) and, hence,

\[
\psi(u) > \frac{K_1}{u},
\]

which contradicts the fact that, according to condition (7), the function \( \psi(u) \) is summable on \([1, \infty)\).

Now assume that

\[
\lim_{u \to \infty} g(u) = \infty,
\]

i.e., for any \( M > 0 \), there exists \( N > 0 \) such that \( g(u) > M \) for all \( u > N \). Then

\[
\int_{1}^{x} \psi(u) du = \int_{1}^{N} \psi(u) du + \int_{N}^{x} g(u) du > \int_{N}^{x} \frac{M}{u} du = K_2 + \int_{N}^{x} \frac{M}{u} du = K_2 + M(\ln x - \ln N).
\]

We again arrive at a contradiction with the condition of the summability of the function \( \psi(u) \) on the interval \([1, \infty)\). It follows from the results presented above that relation (16) is true.
We now prove relation (17). The function $g'(u)$ is summable on $[1, \infty)$, whence
\[
\lim_{u \to \infty} \int_{u/2}^{u} g'(x)dx = 0.
\]

Since, the function $g(u)$ is convex downward for $u \geq b \geq 1$, we conclude that the function $(-g'(u))$ does not increase for $u \geq b$, and, hence,
\[
- \int_{u/2}^{u} g'(x)dx > -\left(u - \frac{u}{2}\right)(\psi(u) + u\psi'(u)) = -\frac{1}{2} \left(u\psi(u) + u^2\psi'(u)\right).
\]

This and relation (16) yield (17).

Taking into account that the function $g(u)$, $u \geq b \geq 1$, is convex downward and using relations (16) and (17), we obtain
\[
\int_{b/\delta}^{\infty} u|d\varphi'(u)| = \int_{b/\delta}^{\infty} u\varphi'(u) = \lim_{u \to \infty} u\varphi'(u) - \frac{b}{\delta}\varphi'\left(b \left(\frac{b}{\delta} \right) + \varphi\left(b \left(\frac{b}{\delta} \right) \right) = \frac{K}{\delta\psi(\delta)}.
\]

Thus,
\[
\int_{0}^{1/2} u|d\varphi'(u)| = O\left(\frac{1}{\delta\psi(\delta)}\right) \quad \text{and} \quad \int_{1/2}^{\infty} |u - 1||d\varphi'(u)| = O\left(\frac{1}{\delta\psi(\delta)}\right) \quad \text{as} \quad \delta \to \infty.
\]

Further, taking into account relation (9) and the inequality
\[
\int_{1}^{\infty} \psi(u)du \leq K,
\]
we get
\[
\int_{0}^{\infty} \frac{\varphi(u)}{u} du = \int_{0}^{\infty} \frac{\varphi(u)}{u} du = \frac{\psi(1)}{\delta\psi(\delta)} + \frac{1}{\delta\psi(\delta)} \int_{1}^{\infty} \psi(u)du = O\left(\frac{1}{\delta\psi(\delta)}\right).
\]

Finally, we estimate the integral
\[
\int_{0}^{1} |\varphi(1 - u) - \varphi(1 + u)| \frac{du}{u}.
\]
For this purpose, we represent this integral as a sum of two integrals:
\[
\int_{0}^{1} |\varphi(1 - u) - \varphi(1 + u)| \frac{du}{u} = \int_{0}^{1 - 1/\delta} |\varphi(1 - u) - \varphi(1 + u)| \frac{du}{u} + \int_{1 - 1/\delta}^{1} |\varphi(1 - u) - \varphi(1 + u)| \frac{du}{u}.
\]
We estimate the first term on the right-hand side of (19) by adding and subtracting the quantity \((-2u)\) under the modulus sign in the integrand. As a result, we get

\[
\int_0^{1-1/\delta} \frac{\varphi(1-u) - \varphi(1+u)}{u} \, du = \int_0^{1-1/\delta} \frac{\varphi(1-u) - \varphi(1+u) - 2u}{u} \, du + O(1). \tag{20}
\]

It follows from (9) that, for \(u \in [0, 1 - 1/\delta]\), we have

\[
1 - u = 1 - \frac{\psi(\delta)}{\psi(\delta(1-u))} \varphi(1-u), \quad 1 + u = 1 - \frac{\psi(\delta)}{\psi(\delta(1+u))} \varphi(1+u).
\]

Then

\[
\int_0^{1-1/\delta} \frac{\varphi(1-u) - \varphi(1+u) - 2u}{u} \, du \leq \int_0^{1-1/\delta} \frac{\varphi(1-u)}{|1 - \frac{\psi(\delta)}{\psi(\delta(1-u))}|} \, du + \int_0^{1-1/\delta} \frac{\varphi(1+u)}{|1 - \frac{\psi(\delta)}{\psi(\delta(1+u))}|} \, du.
\]

Since the function \(\varphi(\cdot)\) satisfies the conditions of Lemma 2 in [8], we have

\[
|\varphi(u)| \leq |\varphi(0)| + |\varphi(1)| + \int_0^{1/2} u |d\varphi'(u)| + \int_{1/2}^{\infty} |u-1| |d\varphi'(u)| := H(\varphi).
\]

Thus,

\[
\int_0^{1-1/\delta} \frac{\varphi(1-u) - \varphi(1+u) - 2u}{u} \, du = H(\varphi)O\left(\int_0^{1-1/\delta} \frac{|\psi(\delta(1-u)) - \psi(\delta)|}{u \psi(\delta(1-u))} \, du + \int_0^{1-1/\delta} \frac{|\psi(\delta(1+u)) - \psi(\delta)|}{u \psi(\delta(1+u))} \, du\right). \tag{21}
\]

Taking into account relation (9) and estimates (18) and using (16), we get

\[
H(\varphi) = O\left(\frac{1}{\delta \psi(\delta)}\right), \quad \delta \to \infty. \tag{22}
\]
It was established in [7] that the following estimates hold for functions $\psi \in \mathcal{M}_0$ as $\delta \to \infty$:

$$\int_0^{1-1/\delta} \frac{|\psi(\delta(1-u)) - \psi(\delta)|}{u \psi(\delta(1-u))} du = O(1), \quad \int_0^{1-1/\delta} \frac{|\psi(\delta(1+u)) - \psi(\delta)|}{u \psi(\delta(1+u))} du = O(1);$$

these estimates are also true for functions $\psi \in \mathcal{M}_C$.

Combining relations (20)–(22), we get

$$\int_0^{1-1/\delta} \frac{|\varphi(1-u) - \varphi(1+u)|}{u} du = O\left(\frac{1}{\delta \psi(\delta)}\right).$$

By analogy, one can easily verify that the same estimate holds for the second term on the right-hand side of (19). Therefore,

$$\int_0^1 \frac{|\varphi(1-u) - \varphi(1+u)|}{u} du = O\left(\frac{1}{\delta \psi(\delta)}\right), \quad \delta \to \infty.$$

Thus, we have established the convergence of integral (13) in the case where $\psi \in \mathcal{M}$, the function $g(u) = u \psi(u)$ is convex downward on $[b, \infty)$, $b \geq 1$, and condition (7) is satisfied. Let us prove the convergence of integral (14). To this end, by virtue of Theorem 1 in [8], it is necessary to estimate the integrals

$$\int_0^{1/2} u|d\psi'(u)|, \quad \int_0^\infty |u - 1||d\psi'(u)|, \quad \int_0^\infty |\sin \frac{\beta \pi}{2} + d\psi(u)| du, \quad \int_0^1 \frac{|\varphi(1-u) - \varphi(1+u)|}{u} du,$$

where $\psi(u)$ is the function given by (10), which is defined and continuous for all $u \geq 0$.

To estimate the first integral in (23), we divide the segment $[0; 1/2]$ into the two parts $[0; 1/\delta]$ and $[1/\delta; 1/2]$. It follows from (10) that

$$\varphi''(u) = -e^{-u} \frac{\psi(1)}{\psi(\delta)} \quad \text{for} \quad u \in \left[0, \frac{1}{\delta}\right].$$

Therefore,

$$\int_0^{1/\delta} u|d\psi'(u)| = \frac{\psi(1)}{\psi(\delta)} \int_0^{1/\delta} u e^{-u} du \leq \frac{\psi(1)}{\psi(\delta)} \int_0^{1/\delta} u du = O\left(\frac{1}{\delta^2 \psi(\delta)}\right).$$

It also follows from relation (10) and properties of the function $\psi \in \mathcal{M}$ that, for $u \geq 1/\delta$, one has

$$|d\psi'(u)| \leq \left\{ \left|\int_0^{\delta^2 \psi''(\delta u)} \psi(\delta) - 2 \left|\int_0^{\delta \psi'(\delta u)} \psi(\delta) - \psi''(u) \frac{\psi(u)}{\psi(\delta)} \right| du, \right.$$
where \( \psi(u) = 1 - e^{-u} - u \). Using the inequalities
\[
|\psi(u)| \leq \frac{u^2}{2}, \quad |\psi(u)| \leq u, \quad |\psi''(u)| \leq 1,
\]
we rewrite relation (26) in the form
\[
|d\psi'(u)| \leq \left\{ u^2 \frac{\delta^2 \psi''(\delta u)}{2\psi(\delta)} + 2u \frac{\delta \psi'(\delta u)}{\psi(\delta)} + \frac{\psi(\delta u)}{\psi(\delta)} \right\} du, \quad \psi \in \mathbb{M}. \tag{27}
\]
Using (27), we obtain the following relation for the first integral in (23) on the segment \([1/\delta, 1/2]\):
\[
\int_{1/\delta}^{1/2} u |d\psi(u)| \leq \frac{1}{\psi(\delta)} \int_{1/\delta}^{1/2} u^3 \frac{\delta^2 \psi''(\delta u)}{2\psi(\delta)} du + \frac{2}{\psi(\delta)} \int_{1/\delta}^{1/2} u^2 \delta \psi'(\delta u) du + \frac{1}{\psi(\delta)} \int_{1/\delta}^{1/2} u \psi(\delta u) du.
\]
Taking the first integral on the right-hand side of the last inequality, we obtain
\[
\int_{1/\delta}^{1/2} u |d\psi(u)| \leq \frac{1}{\psi(\delta)} \left. \frac{u^3}{1/\delta} \frac{\delta^2 \psi'(\delta u)}{2 \psi(\delta)} \right|_{1/\delta}^{1/2} + \frac{7}{\psi(\delta)} \int_{1/\delta}^{1/2} u^2 \delta \psi'(\delta u) du + \frac{1}{\psi(\delta)} \int_{1/\delta}^{1/2} u \psi(\delta u) du. \tag{28}
\]
Further, we use the following statements:

**Proposition 1** [6, p. 161]. A function \( \psi \in \mathbb{M} \) belongs to \( \mathbb{M}_C \) if and only if the quantity
\[
\alpha(t) = \frac{\psi(t)}{t |\psi'(t)|}, \quad \psi'(t) = \psi'(t + 0),
\]
satisfies the condition \( 0 < K_1 \leq \alpha(t) \leq K_2 \) \( \forall t \geq 1 \).

**Proposition 2** [6, p. 175]. A function \( \psi \in \mathbb{M} \) belongs to \( \mathbb{M}_0 \) if and only if, for an arbitrary fixed number \( c > 1 \), there exists a constant \( K \) such that the following inequality holds for all \( t \geq 1 \):
\[
\frac{\psi(t)}{\psi(ct)} \leq K.
\]

Using the conditions of Proposition 1, for \( \psi \in \mathbb{M}_C \) we get
\[
\frac{1}{\psi(\delta)} \int_{1/\delta}^{1/2} u^2 \delta \psi'(\delta u) du \leq \frac{K}{\psi(\delta)} \int_{1/\delta}^{1/2} u \psi(\delta u) du.
\]
Then, using (28) and taking into account Proposition 2 (which is also true for functions \( \psi \in \mathbb{M}_C \)) and the inequality
\[
\int_{1/\delta}^{\delta} u \psi(u) du \geq K,
\]
we obtain
\[ \int_{1/\delta}^{1/2} u|dv(u)| \leq K_1 + K_2 \frac{\int_{1/\delta}^{1/2} u\psi(\delta u)du}{\delta^2 \psi(\delta)} + K_3 \int_{1/\delta}^{1/2} u\psi(\delta u)du \leq \frac{K}{\delta^2 \psi(\delta)} \int_{1/\delta}^{1/2} u\psi(u)du. \] (29)

Combining (25) and (29), we get
\[ \int_{0}^{1/2} u|dv(u)| = O \left( \frac{1}{\delta^2 \psi(\delta)} \int_{1/\delta}^{1/2} u\psi(u)du \right), \quad \delta \to \infty. \] (30)

Let us estimate the second integral in (23). For the function \( \psi(u) = 1 - e^{-u} - u \), we have \( |\psi(u)| \leq u \), \( |\psi'(u)| \leq 1 \), and \( |\psi''(u)| = e^{-u} \). Taking this into account and using (26), we obtain the following relation for \( \delta \geq 2 \):
\[ \int_{1/2}^{\infty} |u - 1||dv'(u)| \leq \int_{1/2}^{\infty} u|dv'(u)| \]
\[ \leq \frac{1}{\psi(\delta)} \int_{1/2}^{\infty} ue^{-u}\psi(\delta u)du + \frac{2\delta}{\psi(\delta)} \int_{1/2}^{\infty} u|\psi'(\delta u)|du + \frac{\delta^2}{\psi(\delta)} \int_{1/2}^{\infty} u^2\psi''(\delta u)du. \] (31)

Let us estimate the first integral on the right-hand side of (31). Since the function \( \psi(\delta u), \delta \geq 2 \), decreases for \( u \in [1/2, \infty] \), taking Proposition 2 into account we get
\[ \frac{1}{\psi(\delta)} \int_{1/2}^{\infty} ue^{-u}\psi(\delta u)du \leq \frac{\psi(\delta/2)}{\psi(\delta)} \int_{1/2}^{\infty} ue^{-u}du = O(1). \] (32)

Integrating the third integral on the right-hand side of inequality (31) by parts and using equality (17) and Propositions 1 and 2, we obtain the following relation for the functions \( \psi(\delta u) \in M_C, u \geq 1/2, \delta \geq 2 \):
\[ \frac{\delta^2}{\psi(\delta)} \int_{1/2}^{\infty} u^2\psi''(\delta u)du = \frac{\delta}{\psi(\delta)} \int_{1/2}^{\infty} u^2d\psi'(\delta u) \]
\[ = \frac{\delta}{\psi(\delta)} \lim_{u \to \infty} u^2\psi'(\delta u) + \frac{(\delta/2)|\psi'(\delta/2)|}{2\psi(\delta)} + \frac{2\delta}{\psi(\delta)} \int_{1/2}^{\infty} u|\psi'(\delta u)|du \]
\[ \leq K_1 + \frac{2\delta}{\psi(\delta)} \int_{1/2}^{\infty} u|\psi'(\delta u)|du. \] (33)
It follows from (31)–(33) that

$$\int_{1/2}^{\infty} |u - 1| d\nu'(u) \leq K_2 + \frac{4\delta}{\psi(\delta)} \int_{1/2}^{\infty} u |\psi'(\delta u)| du.$$ 

Integrating the integral on the right-hand side of the last relation again by parts and using relation (16) and Proposition 2, we obtain

$$\int_{1/2}^{\infty} |u - 1| d\nu'(u) \leq K_3 + \frac{4}{\psi(\delta)} \int_{1/2}^{\infty} \psi(\delta u) du$$

$$\leq K_3 + \frac{2\psi(\delta/2)}{\psi(\delta)} + \frac{4}{\psi(\delta)} \int_{1}^{\infty} \psi(\delta u) du \leq K_4 + \frac{4}{\delta \psi(\delta)} \int_{\delta}^{\infty} \psi(u) du.$$ 

Thus, the following estimate holds as $\delta \to \infty$:

$$\int_{1/2}^{\infty} |u - 1| d\nu'(u) = O \left( 1 + \frac{1}{\delta \psi(\delta)} \int_{\delta}^{\infty} \psi(u) du \right).$$ 

(34)

To estimate the first integral in (24), we divide the interval $[0; \infty)$ into the following three parts: $[0; 1/\delta]$, $[1/\delta; 1]$, and $[1; \infty)$. Taking into account the inequality

$$e^{-u} \leq 1 - u + \frac{u^2}{2}, \quad u \geq 0,$$ 

(35)

we obtain

$$\int_{0}^{1/\delta} \frac{\nu(u)}{u} du = \frac{\psi(1)}{\psi(\delta)} \int_{0}^{1/\delta} (1 - e^{-u} + u) \frac{du}{u} \leq \frac{\psi(1)}{2\psi(\delta)} \int_{0}^{1/\delta} u du = O \left( \frac{1}{\delta^2 \psi(\delta)} \right),$$

$$\int_{1/\delta}^{1} \frac{\nu(u)}{u} du \leq \frac{\psi(\delta u)}{2\psi(\delta)} \int_{1/\delta}^{1} u \psi(u) du = O \left( \frac{1}{\delta^2 \psi(\delta)} \int_{1/\delta}^{\delta} u \psi(u) du \right),$$

$$\int_{1}^{\infty} \frac{\nu(u)}{u} du = \frac{1}{\psi(\delta)} \int_{1}^{\infty} \psi(\delta u) \left( \frac{e^{-u} - 1}{u} + 1 \right) du \leq \frac{1}{\psi(\delta)} \int_{1}^{\infty} \psi(\delta u) du.$$
Hence,

\[
\int_0^{\infty} \frac{|v(u)|}{u} du = O \left( \frac{1}{\delta^2 \psi(\delta)} \int_1^{\delta} u \psi(u) du + \frac{1}{\delta \psi(\delta)} \int_\delta^{\infty} \psi(u) du \right).
\]  

(36)

Let us estimate the second integral in (24). By analogy with the proof of relation (58) in [7], we obtain

\[
\int_0^1 |v(1-u) - v(1+u)| \frac{du}{u} = \int_0^1 |\lambda(1-u) - \lambda(1+u)| \frac{du}{u} + O(\eta(\nu)),
\]  

(37)

where \(\lambda(u) = e^{-u} + u\) and

\[
H(\nu) = |v(0)| + |v(1)| + \int_0^{1/2} u |dv'(u)| + \int_{1/2}^{\infty} |u - 1| |dv'(u)|.
\]

Taking into account relations (10), (30), and (34) and the inequality

\[
\frac{1}{\delta^2 \psi(\delta)} \int_1^{\delta} u \psi(u) du \geq \frac{1}{\delta^2 \psi(\delta)} \delta \psi(\delta) \int_1^{\delta} du \geq K,
\]

we get

\[
H(\nu) = O \left( \frac{1}{\delta^2 \psi(\delta)} \int_1^{\delta} u \psi(u) du + \frac{1}{\delta \psi(\delta)} \int_\delta^{\infty} \psi(u) du \right), \quad \delta \to \infty.
\]

(38)

Since

\[
\int_0^1 |\lambda(1-u) - \lambda(1+u)| \frac{du}{u} \leq K,
\]

(39)

relations (37)–(39) yield the following estimate as \(\delta \to \infty\):

\[
\int_0^1 |v(1-u) - v(1+u)| \frac{du}{u} = O \left( \frac{1}{\delta^2 \psi(\delta)} \int_1^{\delta} u \psi(u) du + \frac{1}{\delta \psi(\delta)} \int_\delta^{\infty} \psi(u) du \right).
\]

(40)

Thus, according to Theorem 1 in [8], integral (14) is also convergent.

Lemma 1 is proved.
Proof of Theorem 1. Lemma 1 states that, under the conditions of Theorem 1, the Fourier transform \( \hat{\tau}(t) \) (3) of the function \( \tau(u) = \varphi(u) + \nu(u) \) is summable on the entire number axis. Then, for any function \( f \in C^\psi_{\beta, \infty} \), equality (6) holds at every point \( x \in \mathbb{R} \).

Using the integral representation (6), we represent quantity (1) in the form

\[
\mathcal{E}\left(C^\psi_{\beta, \infty}; \delta\right) = \sup_{f \in C^\psi_{\beta, \infty}} \left\| \psi(\delta) \int_{-\infty}^{+\infty} f^\psi_\beta \left(x + \frac{t}{\delta}\right) \hat{\tau}(t) dt \right\|_C
\]

\[
= \sup_{f \in C^\psi_{\beta, \infty}} \left\| \psi(\delta) \int_{-\infty}^{+\infty} f^\psi_\beta \left(x + \frac{t}{\delta}\right) \left(\hat{\psi}(t) + \hat{\nu}(t)\right) dt \right\|_C.
\]

Using (14), we obtain

\[
\mathcal{E}\left(C^\psi_{\beta, \infty}; \delta\right) = \sup_{f \in C^\psi_{\beta, \infty}} \left\| \psi(\delta) \int_{-\infty}^{+\infty} f^\psi_\beta \left(x + \frac{t}{\delta}\right) \hat{\phi}(t) dt \right\|_C + O\left(\psi(\delta)A(\nu)\right). \tag{41}
\]

Repeating the arguments of [3], one can easily verify that the Fourier series of the function

\[
f^\phi(x) = \int_{-\infty}^{+\infty} f^\psi_\beta \left(x + \frac{t}{\delta}\right) \hat{\phi}(t) dt
\]

has the form

\[
S[f^\phi] = \sum_{k=1}^{\infty} \frac{1}{\psi(\delta)} \left( a_k \cos kx + b_k \sin kx \right),
\]

where \( a_k \) and \( b_k \) are the Fourier coefficients of the function \( f \). Therefore,

\[
\int_{-\infty}^{+\infty} f^\psi_\beta \left(x + \frac{t}{\delta}\right) \hat{\phi}(t) dt = \frac{1}{\psi(\delta)} f_0^{(1)}(x), \tag{42}
\]

where \( f_0^{(1)}(\cdot) \) is the \((\psi, \beta)\)-derivative of the function \( f(\cdot) \) in the Stepanets sense for \( \psi(t) = 1/t \) and \( \beta = 0 \).

Combining (41) and (42), we obtain

\[
\mathcal{E}\left(C^\psi_{\beta, \infty}; \delta\right) = \frac{1}{\delta} \sup_{f \in C^\psi_{\beta, \infty}} \left\| f_0^{(1)}(x) \right\|_C + O\left(\psi(\delta)A(\nu)\right), \quad \delta \to \infty. \tag{43}
\]
Using inequalities (2.14) and (2.15) from [8] and relations (30), (34), (36), (38), and (40), we obtain the following estimate for the integral $A(v)$:

$$A(v) = O\left(\frac{1}{\delta^2} \psi(\delta) \int_1^\delta u \psi(u) du + \frac{1}{\delta} \int_\delta^\infty \psi(u) du\right), \quad \delta \to \infty.$$  

This and relation (43) yield (8).

Theorem 1 is proved.

Examples of functions satisfying the conditions of Theorem 1 are functions $\psi \in \mathcal{M}$ that have the following form for $t \geq 1$:

$$\psi(t) = \frac{1}{t} \ln^\alpha (t + K), \quad K > 0, \quad \alpha < -1; \quad \psi(t) = \frac{1}{t^r} \ln^\alpha (t + K);$$

$$\psi(t) = \frac{1}{t^r} \arctan t; \quad \psi(t) = \frac{1}{t^r} (K + e^{-t}), \quad r > 1, \quad K > 0, \quad \alpha \in \mathbb{R}.$$  

In the second part of the present paper, we find a solution of the Kolmogorov–Nikol’skii problem for Poisson integrals on the classes $C_{\beta, \infty}^\psi$ of continuous periodic functions in the case where $\psi$ belongs to $\mathcal{M}_\infty$.

**Theorem 2.** If $\psi$ belongs to $\mathcal{M}$, a function $g(u)$ is convex downward for $u \in [b, \infty)$, $b \geq 1$, and

$$\int_1^\infty u^2 \psi(u) du < \infty,$$

then the following asymptotic equality holds as $\delta \to \infty$:

$$\mathcal{E}\left(C_{\beta, \infty}^\psi; P_\delta\right)_C = \frac{1}{\delta} \sup_{f \in C_{\beta, \infty}^\psi} \left\| f_0^{(1)}(x) \right\|_C + O\left(\frac{1}{\delta^2}\right),$$

where $f_0^{(1)}$ is the $(\psi, \beta)$-derivative of the function $f$ for $\psi(t) = 1/t$ and $\beta = 0$.

The proof of Theorem 2 is based on the following auxiliary statement:

**Lemma 2.** Suppose that all conditions of Theorem 2 are satisfied. Then an integral $A(\tau)$ of the form (5) is convergent.

**Proof of Lemma 2.** To establish the convergence of the integral $A(\tau)$ we represent the function $\tau(\cdot)$ (4) as the sum of the functions $\varphi(\cdot)$ and $\nu(\cdot)$ defined by (9) and (10), respectively. We investigate the convergence of integral (13). To this end, we divide the set $(-\infty, \infty)$ into the two subsets $(-\infty, \delta) \cup (\delta, +\infty)$ and $[-\delta, \delta]$.

Let us estimate the integral $A(\varphi)$ for $|t| > \delta$. To this end, we consider the integral

$$\int_0^\infty \varphi(u) \cos \left(ut + \frac{\beta \pi}{2}\right) du$$
on each of the intervals \([0; 1/\delta]\) and \([1/\delta; \infty)\):

\[
\int_{0}^{\infty} \varphi(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du = \left( \int_{0}^{1/\delta} + \int_{1/\delta}^{\infty} \right) \varphi(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du.
\]  
(46)

It follows from (9) that

\[
\varphi(0) = 0, \quad \varphi \left( \frac{1}{\delta} \right) = \frac{\psi(1)}{\delta \psi(\delta)}, \quad \text{and} \quad \varphi'(0) = \varphi' \left( \frac{1}{\delta} - 0 \right) = \frac{\psi(1)}{\psi(\delta)} \quad \text{for} \quad u \in \left[ 0, \frac{1}{\delta} \right).
\]

Integrating the first integral on the right-hand side of equality (46) twice by parts and taking into account that \(\varphi''(u) = 0, \quad u \in [0, 1/\delta]\), we obtain

\[
\int_{0}^{1/\delta} \varphi(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du = \frac{\psi(1)}{t \delta \psi(\delta)} \sin \left( \frac{t}{\delta} + \frac{\beta \pi}{2} \right) + \frac{\psi(1)}{t^2 \psi(\delta)} \left( \cos \left( \frac{t}{\delta} + \frac{\beta \pi}{2} \right) - \cos \frac{\beta \pi}{2} \right).
\]  
(47)

By virtue of the convexity of the function \(g(u)\) and condition (44), relations (16) and (17) are true. For \(u \geq 1/\delta\), we get

\[
\int_{1/\delta}^{\infty} \varphi(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du
\]

\[
= -\frac{\psi(1)}{t \delta \psi(\delta)} \sin \left( \frac{t}{\delta} + \frac{\beta \pi}{2} \right) - \frac{1}{t^2} \left( \frac{\psi(1)}{\psi(\delta)} + \frac{\psi'(1)}{\psi(\delta)} \right) \cos \left( \frac{t}{\delta} + \frac{\beta \pi}{2} \right)
\]

\[
- \frac{1}{t^2} \int_{1/\delta}^{\infty} \varphi''(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du.
\]  
(48)

Combining relations (46)–(48), we obtain

\[
\int_{0}^{\infty} \varphi(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du
\]

\[
= -\frac{1}{t^2 \psi(\delta)} \left( \psi(1) \cos \frac{\beta \pi}{2} + \psi'(1) \cos \left( \frac{t}{\delta} + \frac{\beta \pi}{2} \right) \right) - \frac{1}{t^2} \int_{1/\delta}^{\infty} \varphi''(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du.
\]

Thus,

\[
\left| \int_{0}^{\infty} \varphi(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du \right| \leq \frac{K}{t^2 \psi(\delta)} + \frac{1}{t^2} \int_{1/\delta}^{\infty} |\varphi''(u)| du.
\]  
(49)
Using relation (15) and taking into account that

\[ \lim_{u \to \infty} \psi(u) = 0 \quad \text{and} \quad \lim_{u \to \infty} u \psi'(u) = 0, \]

we get

\[ \frac{1}{t^2} \int_{1/\delta}^{\infty} |\psi''(u)| \, du \leq -\frac{2}{t^2 \psi(\delta)} \int_{1/\delta}^{\infty} d\psi(\delta u) + \frac{\delta}{t^2 \psi(\delta)} \int_{1/\delta}^{\infty} u \psi'(\delta u) = \frac{3\psi(1) - \psi'(1)}{t^2 \psi(\delta)}. \]

Using this relation and (49), we obtain

\[ \left| \int_{0}^{\infty} \varphi(u) \cos \left( ut + \frac{\beta \pi}{2} \right) \, du \right| \leq \frac{K_1}{t^2 \psi(\delta)}, \]

whence

\[ \int_{|t| \geq \delta} \left| \int_{0}^{\infty} \varphi(u) \cos \left( ut + \frac{\beta \pi}{2} \right) \, du \right| \, dt \leq \frac{2K_1}{\delta \psi(\delta)}. \] (50)

Let us estimate the integral \( A(\psi) \) on the segment \([-\delta, \delta]\). Since condition (44) is satisfied, we have

\[ \int_{-\delta}^{\delta} \left| \int_{0}^{\infty} \varphi(u) \cos \left( ut + \frac{\beta \pi}{2} \right) \, du \right| \, dt \]

\[ \leq 2\delta \int_{0}^{\infty} |\varphi(u)| \, du = \frac{\psi(1)}{\delta \psi(\delta)} + \frac{2\delta}{\psi(\delta)} \int_{1/\delta}^{\infty} u \psi(\delta u) \, du \]

\[ = \frac{\psi(1)}{\delta \psi(\delta)} + \frac{2}{\delta \psi(\delta)} \int_{1/\delta}^{\infty} u \psi(u) \, du \leq \frac{K_2}{\delta \psi(\delta)}. \] (51)

Using relations (50) and (51), we conclude that the following estimate holds as \( \delta \to \infty \):

\[ A(\psi) = O\left( \frac{1}{\delta \psi(\delta)} \right). \]

Thus, the transform \( \hat{\psi}(t) \) (11) is summable on the entire number axis.

We now establish the convergence of the integral \( A(v) \) [see (14)], where \( \hat{v}(t) \) is the transform (12) of the function \( v(\cdot) \) defined by relation (10). To this end, we divide the set \((-\infty, \infty)\) into the two parts \([-\delta, \delta]\) and \(|t| > \delta\) so that
Let us estimate the integral

\[ I_1 = \frac{1}{\pi} \int_{-\delta}^{\delta} \int_{0}^{1/\delta} v(u) \cos \left( ut + \frac{\beta \pi}{2} \right) \, du \, dt. \]

We have

\[ I_1 \geq \frac{1}{\pi} \int_{-\delta}^{\delta} \int_{0}^{1/\delta} v(u) \cos \left( ut + \frac{\beta \pi}{2} \right) \, du \, dt \geq \frac{1}{\pi} \int_{-\delta}^{\delta} \int_{0}^{1/\delta} |v(u)| \, du \, dt. \]

Taking inequality (35) into account, we obtain

\[ \frac{1}{\pi} \int_{-\delta}^{\delta} \int_{0}^{1/\delta} v(u) \cos \left( ut + \frac{\beta \pi}{2} \right) \, du \, dt \leq \frac{1}{\pi} \int_{-\delta}^{\delta} \int_{0}^{1/\delta} |v(u)| \, du \, dt \leq \frac{2\delta \psi(1)}{\pi \psi(\delta)} \int_{0}^{1/\delta} (e^{-u} + u - 1) \, du \leq \frac{\psi(1)}{3\pi \delta^2 \psi(\delta)}. \]

According to the conditions of Lemma 2, we have

\[ \int_{1}^{\infty} u^2 \psi(u) \, du < \infty. \]

Using inequality (35) once again, we obtain the following estimate for the second integral on the right-hand side of (53):

\[ \frac{1}{\pi} \int_{-\delta}^{\delta} \int_{1/\delta}^{\infty} v(u) \cos \left( ut + \frac{\beta \pi}{2} \right) \, du \, dt \leq \frac{1}{\pi} \int_{-\delta}^{\delta} \int_{1/\delta}^{\infty} |v(u)| \, du \, dt \leq \frac{2\delta \psi(1)}{\pi \psi(\delta)} \int_{1/\delta}^{\infty} (e^{-u} + u - 1) \psi(\delta u) \, du \]

\[ \leq \frac{\delta}{\pi \psi(\delta)} \int_{1/\delta}^{\infty} u^2 \psi(\delta u) \, du \leq \frac{K}{\pi \delta^2 \psi(\delta)}. \]

It follows from relations (53)–(55) that

\[ I_1 = O \left( \frac{1}{\delta^2 \psi(\delta)} \right), \quad \delta \to \infty. \]
Let us estimate the integral

\[ I_2 = \frac{1}{\pi} \int_{|t| > \delta} \int_0^\infty v(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du \, dt. \]

Integrating twice by parts and taking into account that \( v(0) = 0 \) and \( v'(0) = 0 \), we get

\[ \int_0^{1/\delta} v(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du \]

\[ = \frac{1}{t} v \left( \frac{1}{\delta} \right) \sin \left( \frac{t}{\delta} + \frac{\beta \pi}{2} \right) + \frac{1}{t^2} v' \left( \frac{1}{\delta} \right) \cos \left( \frac{t}{\delta} + \frac{\beta \pi}{2} \right) - \frac{1}{t^2} \int_0^{1/\delta} v''(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du. \] (57)

Taking into account that

\[ \lim_{u \to \infty} v(u) = 0 \quad \text{and} \quad \lim_{u \to \infty} v'(u) = 0, \]

which follows from (16) and (17), we obtain

\[ \int_{1/\delta}^\infty v(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du \]

\[ = -\frac{1}{t} v \left( \frac{1}{\delta} \right) \sin \left( \frac{t}{\delta} + \frac{\beta \pi}{2} \right) - \frac{1}{t^2} v' \left( \frac{1}{\delta} \right) \cos \left( \frac{t}{\delta} + \frac{\beta \pi}{2} \right) - \frac{1}{t^2} \int_{1/\delta}^\infty v''(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du. \] (58)

Combining (57) and (58), we get

\[ \int_0^\infty v(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du \]

\[ = \frac{1}{t^2} \left( \frac{1}{\delta} + e^{-1/\delta} - 1 \right) \frac{\delta \psi' \left( \frac{1}{\delta} \right)}{\psi \left( \frac{1}{\delta} \right)} \cos \left( \frac{t}{\delta} + \frac{\beta \pi}{2} \right) - \frac{1}{t^2} \int_0^{1/\delta} + \int_{1/\delta}^\infty v''(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du. \]

Using inequality (35), we obtain

\[ \int_0^\infty v(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du \leq \frac{1}{t^2} \left( \frac{K}{\delta \psi \left( \frac{1}{\delta} \right)} + \int_0^{1/\delta} \left| v''(u) \right| du + \int_{1/\delta}^\infty \left| v''(u) \right| du \right). \] (59)
Since
\[ v''(u) = -e^{-u} \frac{\psi(1)}{\psi(\delta)} \quad \text{for} \quad u \in \left[ 0, \frac{1}{\delta} \right), \]
we get
\[ \frac{1}{t^2} \int_{0}^{1/\delta} |v''(u)| du = \frac{\psi(1)}{t^2\psi(\delta)} \left( \int_{0}^{1/\delta} e^{-u} du \right) \leq \frac{\psi(1)}{t^2\delta \psi(\delta)}. \quad (60) \]

For the estimation of the second integral on the right-hand side of (59), we use relations (27), (16), and (17). As a result, we obtain
\[ \frac{1}{t^2} \int_{1/\delta}^{\infty} |v''(u)| du \leq \frac{1}{t^2\psi(\delta)} \left( \int_{1/\delta}^{\infty} \psi(\delta u) du - 2 \int_{1/\delta}^{\infty} u d\psi(\delta u) + \frac{\delta}{2} \int_{1/\delta}^{\infty} u^2 d\psi'(\delta u) \right) \]
\[ = \frac{1}{t^2\psi(\delta)} \left( \lim_{u \to \infty} u \psi(\delta u) - \frac{\psi(1)}{\delta} - \int_{1/\delta}^{\infty} \psi(\delta u) du \right) \]
\[ + \frac{\delta}{2} \left( \lim_{u \to \infty} u^2 \psi'(\delta u) - \frac{\psi'(1)}{\delta^2} - \int_{1/\delta}^{\infty} u d\psi(\delta u) \right) \]
\[ = \frac{1}{t^2\psi(\delta)} \left( 4 \int_{1/\delta}^{\infty} \psi(\delta u) du + \frac{3\psi(1)}{\delta} - \frac{\psi'(1)}{2\delta} \right). \]

Since
\[ \int_{1}^{\infty} \psi(u) du < \infty, \]
we have
\[ \frac{1}{t^2} \int_{1/\delta}^{\infty} |v''(u)| du \leq \frac{K}{t^2\delta \psi(\delta)}. \quad (61) \]

It follows from relations (59)–(61) that
\[ \left| \int_{0}^{\infty} v(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du \right| \leq \frac{K_1}{t^2\delta \psi(\delta)}. \]
Then the following relation holds as \( \delta \to \infty \):

\[
I_2 = \frac{1}{\pi} \int_{|r|>\delta} \left| \int_0^\infty v(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du \right| dt = O \left( \frac{1}{\delta^2 \psi(\delta)} \right). \tag{62}
\]

Combining relations (52), (56), and (62), we get

\[
A(\nu) = O \left( \frac{1}{\delta^2 \psi(\delta)} \right), \quad \delta \to \infty. \tag{63}
\]

Lemma 2 is proved.

**Proof of Theorem 2.** Lemma 2 states that integrals (13) and (14) are summable under the conditions of Theorem 2. Therefore, using relation (43) and taking estimate (63) into account, we obtain equality (45).

Theorem 2 is proved.

Examples of functions satisfying the conditions of Theorem 2 are functions \( \psi \in \mathfrak{M} \) that have the following form for \( t \geq 1 \):

\[
\psi(t) = \frac{\ln^\alpha (t + K)}{t^r}, \quad \psi(t) = \frac{1}{t^r} (K + e^{-t}), \quad r > 3, \quad K > 0, \quad \alpha \in \mathbb{R},
\]

\[
\psi(t) = t^r e^{-Kt^\alpha}, \quad \psi(t) = \ln^r (t + e) e^{-Kt^\alpha}, \quad K > 0, \quad \alpha > 0, \quad r \in \mathbb{R}.
\]

Assume that a function \( \mu(\cdot) \) is associated with a function \( \psi \in \mathfrak{M} \) by relation (2). Theorem 2 yields the following corollary:

**Corollary 1.** If \( \psi \) belongs to \( \mathfrak{M}_\infty \), the function \( g(u) \) is convex downward for \( u \in [b, \infty) \), \( b \geq 1 \), and

\[
\lim_{t \to \infty} \mu(\psi; t) = \infty, \tag{64}
\]

then the asymptotic equality (45) holds as \( \delta \to \infty \).

**Proof.** It suffices to verify that condition (64) guarantees the convergence of the integral

\[
\int_1^\infty u^2 \psi(u) du.
\]

It follows from relations (12.24) in [6, p. 164] that the following inequality holds for any function \( \psi \in \mathfrak{M} \):

\[
\frac{\psi(t)}{|\psi'(t)|} \leq 2 (\eta(t) - t) \quad \forall t \geq 1. \tag{65}
\]
In view of (65), for any $r \geq 0$ one has

$$(tr'\psi(t))' = rt^{r-1}\psi(t) - t^r|\psi'(t)| \leq t^r|\psi'(t)| \left(2r\frac{\eta(t)-t}{t} - 1\right). \tag{66}$$

According to (64), the value $(\eta(t)-t)/t$ tends to zero as $t \to \infty$. Using relations (66), we conclude that, for any $r \geq 0$, there exists a number $t_0 = t_0(r, \psi)$ such that the function $t^r\psi(t)$ does not increase for $t > t_0$. Then

$$\int_1^\infty u^2\psi(u)du = \int_1^\infty \frac{u^r\psi(u)}{u^2}du \leq K \int_1^\infty \frac{du}{u^2} < \infty.$$

Thus, all conditions of Theorem 2 are satisfied. Therefore, equality (45) is true.

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